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Abstract. In previous papers [1-2], we have presented hyperbolic governing equations and jump conditions for barotropic fluid mixtures. Now we extend our results to the most general case of two-fluid conservative mixtures taking into account the entropies of components. We obtain governing equations for each component of the medium. This is not a system of conservation laws. Nevertheless, using Hamilton's principle we are able to obtain a complete set of Rankine-Hugoniot conditions. In particular, for the gas dynamics they coincide with classical jump conditions of conservation of momentum and energy. For the two-fluid case, the jump relations do not involve the conservation of the total momentum and the total energy.

Sommario. In precedenti lavori [1-2] sono state dedotte equazioni di governo iperboliche e condizioni di salto per miscele fluide barotropiche. In questo lavoro tali risultati sono estesi al caso più generale di miscele di due fluidi conservative, tenendo conto delle entropie dei componenti. Si ottengono equazioni di governo per ciascun componente della miscela. Pur non essendo queste un sistema di leggi di conservazione, usando il principio di Hamilton si ottiene un insieme completo di condizioni di salto di Rankine-Hugoniot. In particolare per la gasdinamica, queste coincidono con le condizioni di salto classiche per la conservazione del momento e dell'energia. Nel caso dei due fluidi, le condizioni di salto non coinvolgono la conservazione del momento e dell'energia totali.

Key words: Hamilton's principle, Jump conditions, Multiphase flows.[1]

1. Introduction

Hamilton's principle is a well-known method to obtain the equations of motion in conservative fluid mechanics (see, for example, [3-6]). It is less known that this variational method is suitable to obtain the Rankine-Hugoniot conditions through a surface of discontinuity [7-9]. The variations of Hamilton's action are constructed in terms of virtual motions of continua. The virtual motions may be defined both in Lagrangian and Eulerian coordinates [3,10]. Such virtual motions yield the governing equations in different but equivalent forms. However the shock conditions are not equivalent. For example, for the gas dynamics, Hamilton's principle in the Lagrangian coordinates yields the conservation of momentum and energy. In the Eulerian coordinates, we obtain only the conservation of energy and the conservation of the tangential part of the velocity field [7,8]. Here we use variations in the Lagrangian coordinates.

[1] *Extended version of the paper: "Meccanica 34: 39-47, 1999".*

We assume that Hamilton's action is defined with the help of a Lagrangian function which is the difference between the kinetic energy and a potential depending on the densities, the entropies and the relative velocity of the components of the mixture. This potential can be interpreted as a Legendre transformation of the internal energy [1-2,11].

In [1-2,11], we have considered only the case of molecular mixtures. The heterogeneous fluid, when each component occupies only a part of the mixture volume, can be described by the same system if one of the component is incompressible. Indeed, if the phase "1" is incompressible, the average density ρ_1 is related with the volume concentration of component φ_1 by $\rho_1 = \rho_{10}\varphi_1$, where $\rho_{10} =$ const is the physical density of the phase "1". Hence, the knowledge of the average densities gives the volume concentrations and the physical densities. In the present paper, we do not distinguish these two cases. We shall call both cases "two-fluid mixtures".

It is well known (see for example Stewart and Wendroff [12]) that the governing equations of two-fluid mixtures are not generally in a divergence form. In this case we may not obtain the shock conditions for the system. Moreover, the system is often non-hyperbolic, which means the ill-posedness of the Cauchy problem. The hyperbolic two-fluid models were constructed by many authors (see for example [13]). The problem to obtain the Rankine-Hugoniot conditions was an open question. This is the aim of our paper. By using Hamilton's principle in non-isentropic case we obtain the governing equations for each component and a complete set of Rankine-Hugoniot conditions generalizing those obtained in [1-2] for barotropic motions. To present the basic ideas, we consider first in section 2 the one-velocity case and extend this approach in sections 3 and 4 to the two-fluid mixtures.

Let us use asterisk "$*$" to denote *conjugate* (or *transpose*) mappings or covectors (line vectors). For any vectors $\mathbf{a}, \mathbf{b}$ we shall use the notation $\mathbf{a}^*\mathbf{b}$ for their *scalar product* (the line vector is multiplied by the column vector) and $\mathbf{a}\mathbf{b}^*$ for their *tensor product* (the column vector is multiplied by the line vector). The product of a mapping A by a vector $\mathbf{a}$ is denoted by $A\,\mathbf{a}$. Notation $\mathbf{b}^*\,A$ means covector $\mathbf{c}^*$ defined by the rule $\mathbf{c}^* = (A^*\,\mathbf{b}\,)^*$. The divergence of a linear transformation A is the covector $\operatorname{div}A$ such that, for any constant vector $\mathbf{a}$,

$$\operatorname{div}(A)\,\mathbf{a} \;=\; \operatorname{div}\,(A\,\mathbf{a}\,).$$

Let A be any linear mapping defined on Ω_0 and $B = \dfrac{\partial \mathbf{z}}{\partial \mathbf{Z}}$ be the Jacobian matrix associated with the change of variables $\mathbf{z} = \mathbf{M}(\mathbf{Z})$, $\mathbf{z}$ belongs to Ω. Then,

$$\operatorname{div}_0 A \;=\; \det B\,\operatorname{div}\left(\frac{B}{\det B}\,A\right), \tag{1.1}$$

where div_0 (div) means the divergence operator in Ω_0 (Ω). Equation (1.1) plays an important role.

The identical transformation is denoted by I, and the gradient line (column) operator by ∇ (∇^*). For divergence and gradient operators in time-space we use respectively symbols Div and Grad.

The elements of the matrix A are denoted be a^i_j where i means lines and j columns. The elements of the inverse matrix A^{-1} are denoted by $\bar{a}^i_j$. If $f(A)$ is a scalar function of A, the matrix $\dfrac{\partial f}{\partial A}$ is defined by the formula

$$\left(\frac{\partial f}{\partial A}\right)^j_i = \frac{\partial f}{\partial a^i_j}.$$

The repeated latin indices imply summation. Index $\alpha = 1, 2$ refers to the parameters of components: densities ρ_α, velocities $\mathbf{u}_\alpha$, etc.

2. One velocity fluid

The consequences of this section are well known. We obtain the classical governing equations and the Rankine-Hugoniot conditions for the gas dynamics. Nevertheless, the presented method is universal and is extended for two-fluid mixtures in the following sections.

Let $\mathbf{z} = \begin{pmatrix} t \\ \mathbf{x} \end{pmatrix}$ be Eulerian coordinates of a particle and $\mathcal{D}(t)$ a volume of the physical space occupied by a fluid at time t. When t belongs to the finite interval $[t_0, t_1]$, $\mathcal{D}(t)$ generates a four-dimensional domain Ω in the time-space. A particle is labelled by its position $\mathbf{X}$ in a reference space $\mathcal{D}_0$. For example, if $\mathcal{D}(t)$ consists always of the same particles $\mathcal{D}_0 = \mathcal{D}(t_0)$ and we can define the motion of the continuum as a diffeomorphism from $\mathcal{D}(t_0)$ into $\mathcal{D}(t)$:

$$\mathbf{x} = \varphi_t(\mathbf{X}). \tag{2.1}$$

The motion (2.1) of the fluid is generalized in the following parametric form

$$\left\{\begin{array}{l} t \ = g(\lambda, \mathbf{X}) \\ \mathbf{x} = \phi(\lambda, \mathbf{X}) \end{array}\right. \qquad \text{or} \qquad \mathbf{z} = \mathbf{M}(\mathbf{Z}), \tag{2.2}$$

where $\mathbf{Z} = \begin{pmatrix} \lambda \\ \mathbf{X} \end{pmatrix}$ belongs to a reference space denoted Ω_0 and $\mathbf{M}$ is a diffeomorphism from a reference space Ω_0 into the time-space Ω occupied by the medium.
Equations (2.2) lead to the following expressions for the differentials dt and $d\mathbf{x}$,

$$\begin{pmatrix} dt \\ d\mathbf{x} \end{pmatrix} = B \begin{pmatrix} d\lambda \\ d\mathbf{x} \end{pmatrix}, \tag{2.3}$$

where

$$B = \frac{\partial \mathbf{M}}{\partial \mathbf{Z}} = \begin{bmatrix} \dfrac{\partial g}{\partial \lambda}\ , & \dfrac{\partial g}{\partial \mathbf{X}} \\ \dfrac{\partial \phi}{\partial \lambda}\ , & \dfrac{\partial \phi}{\partial \mathbf{X}} \end{bmatrix}. \tag{2.4}$$

In an explicit form, we obtain from (2.3)-(2.4)

$$\left\{\begin{aligned} dt &= \frac{\partial g}{\partial \lambda}\, d\lambda + \frac{\partial g}{\partial \mathbf{X}}\, d\mathbf{X}, \\ d\mathbf{x} &= \frac{\partial \phi}{\partial \lambda}\, d\lambda + \frac{\partial \phi}{\partial \mathbf{X}}\, d\mathbf{X}. \end{aligned}\right. \tag{2.5}$$

Eliminating $d\lambda$ from the first equation of (2.5) and substituting into the second, we obtain

$$d\mathbf{x} = \mathbf{u}\, dt + F d\mathbf{X},$$

where the velocity $\mathbf{u}$ and the deformation gradient F are defined by

$$\mathbf{u} = \frac{\partial \phi}{\partial \lambda}\ \Big(\frac{\partial g}{\partial \lambda}\Big)^{-1}\ ,\ F\ =\ \frac{\partial \phi}{\partial \mathbf{X}}\ -\ \frac{\partial \phi}{\partial \lambda}\, \frac{\partial g}{\partial \mathbf{X}} \Big(\frac{\partial g}{\partial \lambda}\Big)^{-1}. \tag{2.6}$$

Let

$$\left\{\begin{array}{l} t = G(\lambda, \mathbf{X}, \varepsilon) \\ \\ \mathbf{x} = \Phi(\lambda, \mathbf{X}, \varepsilon) \end{array}\right. \qquad \text{or} \qquad \mathbf{z} = \mathbf{M}_\varepsilon(\mathbf{Z}), \tag{2.7}$$

where ε is a scalar defined in the vicinity of zero, be a one-parameter family of virtual motions of the medium such that

$$\mathbf{M}_0(\mathbf{Z}) = \mathbf{M}(\mathbf{Z}).$$

We define the virtual displacement $\boldsymbol{\zeta} = (\tau, \boldsymbol{\xi})$ associated with the virtual motion (2.7):

$$\begin{array}{c} \tau = \dfrac{\partial G}{\partial \varepsilon}(\lambda, \mathbf{X}, 0)\ ,\ \boldsymbol{\xi} = \dfrac{\partial \Phi}{\partial \varepsilon}(\lambda, \mathbf{X}, 0) \\ \text{or,} \qquad \boldsymbol{\zeta} = \dfrac{\partial \mathbf{M}_\varepsilon}{\partial \varepsilon}\ (\mathbf{Z})|_{\varepsilon=0} \end{array} . \tag{2.8}$$

From the mathematical point of view, spaces Ω_0 and Ω play a symmetric role. From the physical point of view they are not symmetric: the tensorial quantities (thermodynamic or mechanical) are defined either on Ω_0 or on Ω. Their image in the dual space depends on the motion of the medium. For example, the potential of body forces Π is defined on Ω and the entropy s is defined on Ω_0.
Let us consider any tensorial quantity represented in the form

$$(t, \mathbf{x})\ \in\ \Omega \longrightarrow f(t, \mathbf{x}).$$

The tensorial quantity associated with the varied motion is

$$\tilde{f}\ (\lambda, \mathbf{X}, \varepsilon) = f\Big(G(\lambda, \mathbf{X}, \varepsilon)\ ,\ \Phi(\lambda, \mathbf{X}, \varepsilon)\Big) = f\Big(\mathbf{M}_\varepsilon(\mathbf{Z})\Big).$$

We define the variation of f by

$$\delta f = \frac{\partial \tilde{f}}{\partial \varepsilon}\ (\lambda, \mathbf{X}, 0).$$

For a tensorial quantity represented in the form

$$(\lambda, \mathbf{X})\ \in\ \Omega_0 \longrightarrow h(\lambda, \mathbf{X}),$$

the tensorial quantity associated with the varied motion is unchanged and

$$\delta h =\ 0$$

Let the Lagrangian of the medium be defined in the form

$$L = L\Big(\mathbf{z},\ \frac{\partial \mathbf{M}}{\partial \mathbf{Z}}\ ,\ \mathbf{Z}\Big) = L(\mathbf{z}, B, \mathbf{Z})\ .$$

This expression contains the gas dynamics model where the Lagrangian is [3]

$$\begin{array}{rl} L = & \frac{1}{2}\ \rho\Big(1 + |\mathbf{u}|^2\Big) - \varepsilon(\rho, s) - \rho\Pi(\mathbf{z}) \\ = & \dfrac{1}{2}\rho\mathbf{V}^*\mathbf{V} - \varepsilon(\rho, s) - \rho\Pi(\mathbf{z}) \end{array} . \tag{2.9}$$

Here $\mathbf{V} = \begin{pmatrix} 1 \\ \mathbf{u} \end{pmatrix}$ is the time-space velocity, $\Pi(\mathbf{z})$ is the external force potential, ρ is the density defined by

$$\rho\ \det\ F = \rho_0(\mathbf{X}),$$

and s is the entropy per unit mass defined by

$$s = s_0(\mathbf{Z}).$$

It is not necessary to assume that s_0 is a function of $\mathbf{X}$ only. This property will be a consequence of the variational principle (see formula (D.1) in Appendix D).
The Hamilton action is:

$$a = \int_\Omega L(\mathbf{z}, B, \mathbf{Z})\ d\Omega. \tag{2.10}$$

For the gas dynamics we obtain from (2.4), (2.6):

$$\mathbf{V} = \begin{pmatrix} 1 \\ \mathbf{u} \end{pmatrix} = \frac{B\boldsymbol{\ell}}{\mu}, \tag{2.11}$$

$$\text{where} \quad \boldsymbol{\ell}^* = (1,0,0,0) \quad \text{and} \quad \mu = \boldsymbol{\ell}^* B\boldsymbol{\ell} = b_1^1 = \frac{\partial g}{\partial \lambda}.$$

Consequently,

$$\frac{1}{2}\left(1 + |\mathbf{u}|^2\right) = \frac{1}{2}\ \frac{\boldsymbol{\ell}^* B^* B\ \boldsymbol{\ell}}{\mu^2}. \tag{2.12}$$

Moreover,

$$\rho \ = \ \frac{\mu}{\det B}\ \rho_0(\mathbf{X}). \tag{2.13}$$

In the Lagrangian coordinates Hamilton's action (2.10) is

$$a = \int_{\Omega_0} L(\mathbf{z}, B, \mathbf{Z}) \det B \ d\Omega_0,$$

and the varied action is

$$a(\varepsilon) = \int_{\Omega_0} L\Big(\mathbf{M}_\varepsilon(\mathbf{Z}),\ \frac{\partial \mathbf{M}_\varepsilon(\mathbf{Z})}{\partial \mathbf{Z}},\ \mathbf{Z}\Big)\ \det\left(\frac{\partial \mathbf{M}_\varepsilon(\mathbf{Z})}{\partial \mathbf{Z}}\right)\ d\Omega_0.$$

Let $T(\Omega)$ be the tangent bundle of Ω.

The Hamilton principle is:

For any continuous virtual displacement $\boldsymbol{\zeta}$ belonging to $T(\Omega)$ such that $\boldsymbol{\zeta} = 0$ on $T(\partial\Omega)$,

$$\delta a = \frac{da}{d\varepsilon}\ |_{\varepsilon\ =\ 0} = 0.$$

Consequently,

$$\delta a = \int_{\Omega_0} \Big\{ \det B\ \frac{\partial L}{\partial \mathbf{z}}\boldsymbol{\zeta} + \det B\ tr\Big(\ \frac{\partial L}{\partial B}\delta B\Big) + L\ \delta(\det B)\Big\} d\Omega_0.$$

The Euler-Jacobi identity

$$\delta \det B = tr\Big(\ \frac{\partial \det B}{\partial B}\ \ \delta B\Big) = tr\ (B^{-1} \det B \delta B)\ = \det B\ tr(B^{-1}\ \ \delta B)$$

and the relation issued from definitions (2.4), (2.8)

$$\delta B = \frac{\partial \boldsymbol{\zeta}}{\partial \mathbf{Z}}\ = \ \frac{\partial \boldsymbol{\zeta}}{\partial \mathbf{z}}\ B$$

yield

$$\delta a = \int_{\Omega} \Big\{ \mathbf{S}^*\boldsymbol{\zeta}\ +\ tr\ \left(T\ \frac{\partial \boldsymbol{\zeta}}{\partial \mathbf{z}}\ \right) \Big\}\ d\Omega,$$

where

$$\mathbf{S}^*\ =\ \frac{\partial L}{\partial \mathbf{z}} \quad \text{and} \quad T\ =\ L\ I\ +\ B\ \frac{\partial L}{\partial B}.$$

The Gauss-Ostrogradskii formula involves

$$\delta a\ =\ \int_{\Omega} (\,\mathbf{S}^*\ -\ \mathrm{Div}\ T\,)\,\boldsymbol{\zeta}\ d\ \Omega\ +\ \int_{\partial\Omega} \mathbf{N}^* T\boldsymbol{\zeta}\ d\omega,$$

where $\mathbf{N}^*$ is the external normal to $\partial\Omega$ and $d\omega$ is the local measure of $\partial\Omega$.

If the motion is continuous on Ω and $\boldsymbol{\zeta} = 0$ on $\partial\Omega$, we get

$$\delta a = \int_{\Omega} (\mathbf{S}^* - \mathrm{Div}\, T)\, \boldsymbol{\zeta}\, d\,\Omega.$$

Consequently, the governing equations are

$$\mathbf{S}^* - \mathrm{div}\, T = 0. \tag{2.14}$$

In Appendix A we verify that (2.14) corresponds to classical momentum and energy equations. If there exists a surface Σ of discontinuity of B separating Ω into two parts Ω_1 and Ω_2 we get

$$\delta a = \int_{\Omega_1} (\mathbf{S}^* - \mathrm{Div}\, T)\, \boldsymbol{\zeta}\, d\,\Omega_1 + \int_{\Omega_2} (\mathbf{S}^* - \mathrm{Div}\, T)\, \boldsymbol{\zeta}\, d\,\Omega_1 + \int_{\Sigma} \mathbf{N}^*[T]\boldsymbol{\zeta}\, d\,\omega,$$

where $[T] = T_1 - T_2$ denotes the jump of T Σ.
Consequently the fundamental lemma of calculus of variations involves the Rankine-Hugoniot conditions

$$\mathbf{N}^*[T] = 0. \tag{2.15}$$

Because $\mathbf{N}^*$ is collinear to $[-D_n\ ,\ \mathbf{n}^*]$, where D_n is the normal velocity of Σ and $\mathbf{n}$ is the normal unit space vector, relations (2.15) are the classical Rankine-Hugoniot conditions representing the conservation of momentum and energy through the shock (see Appendix A and [7]).

3. Two-fluid models: General calculations

We shall study now two-fluid motions, the method being extended to any number of components. We generalize the representation of the motion (2.2) considering the motion of a two-fluid mixture as two diffeomorphisms [14]

$$\mathbf{z} = \mathbf{M}_\alpha\, (\mathbf{Z}_\alpha),$$

where $\mathbf{Z}_\alpha = \begin{bmatrix} \lambda_\alpha \\ \mathbf{X}_\alpha \end{bmatrix}$ belongs to a reference space Ω_α associated with the $\alpha-$th component. The Jacobian matrix is defined by the formula

$$B_\alpha = \frac{\partial \mathbf{M}_\alpha}{\partial \mathbf{Z}_\alpha}\, (\mathbf{Z}_\alpha).$$

The velocity $\mathbf{u}_\alpha$ and the deformation gradient F_α are defined similarly to (2.6). Two one-parameter families of virtual motions are associated with the two diffeomorphisms

$$\begin{cases} \mathbf{z} = \mathbf{M}_{1,\varepsilon_1}\, (\mathbf{Z}_1) \\ \mathbf{z} = \mathbf{M}_2\, (\mathbf{Z}_2) \end{cases} \tag{3.1}$$

such that $\mathbf{M}_{1,0}\, (\mathbf{Z}_1) = \mathbf{M}_1\, (\mathbf{Z}_1)$, and

$$\begin{cases} \mathbf{z} = \mathbf{M}_1\, (\mathbf{Z}_1) \\ \mathbf{z} = \mathbf{M}_{2,\varepsilon_2}\, (\mathbf{Z}_2) \end{cases} \tag{3.2}$$

such that $\mathbf{M}_{2,0}\, (\mathbf{Z}_2) = \mathbf{M}_2\, (\mathbf{Z}_2)$.
The two families extend the concept of virtual motion defined in section 2. Consider the family (3.1), but all the consequences are the same for the family (3.2). We define the virtual displacement of the first component by the relation

$$\boldsymbol{\zeta}_1 = \frac{\partial \mathbf{M}_{1,\varepsilon_1}}{\partial \varepsilon_1}(\mathbf{Z}_1)\, |_{\varepsilon_1 = 0}\, .$$

Virtual motion (3.1) generates a displacement of component "2"

$$\mathbf{M}_{1,\varepsilon_1}\ (\mathbf{Z}_1)\ =\ \mathbf{M}_2\ (\mathbf{Z}_2),$$

which defines $\mathbf{Z}_2$ as a function of $\mathbf{Z}_1$ and ε_1. Taking the derivative with respect to ε_1 and denoting

$$\delta_1 \mathbf{Z}_2\ =\ \frac{\partial \mathbf{Z}_2}{\partial \varepsilon_1}\ |_{\varepsilon_1=0}\ ,$$

we obtain

$$\delta_1 \mathbf{Z}_2\ =\ B_2^{-1} \boldsymbol{\zeta}_1. \tag{3.3}$$

Let us consider any tensorial quantity f in Eulerian coordinates:

$$\mathbf{z}\ \in\ \Omega\ \longrightarrow\ f(\mathbf{z}).$$

The tensorial quantity associated with the varied motion is then

$$\widetilde{f}\ (\mathbf{Z}_1, \varepsilon_1)\ =\ f\Big(\mathbf{M}_{1,\varepsilon_1}(\mathbf{Z}_1)\Big)$$

and consequently $\delta_1 f\ =\ \dfrac{\partial \widetilde{f}}{\partial \varepsilon_1}(\mathbf{Z}_1, 0)$ is the variation of f.
Let us consider the Lagrangian of the medium in the representation

$$L\ =\ L(\mathbf{z}, B_1, B_2, \mathbf{Z}_1, \mathbf{Z}_2). \tag{3.4}$$

For example, the Lagrangian of a two-fluid mixture is [1-2, 4-6, 11]:

$$L\ =\ \frac{1}{2}\,\rho_1\ (1+|\mathbf{u}_1|^2) + \frac{1}{2}\,\rho_2\ (1+|\mathbf{u}_2|^2) - W(\rho_1, \rho_2, s_1, s_2, \mathbf{u}_2 - \mathbf{u}_1) - \rho\, \Pi(\mathbf{z}), \tag{3.5}$$

where ρ_α is the density of the $\alpha - \mathrm{th}$ component defined by

$$\rho_\alpha\ \det\ F_\alpha\ =\ \rho_{0,\alpha}(\mathbf{X}_\alpha),$$

s_α is the entropy per unit mass of the $\alpha - \mathrm{th}$ component defined by the relation

$$s_\alpha\ =\ s_{0,\alpha}(\mathbf{Z}_\alpha),$$

and $\Pi(\mathbf{z})$ is the potential of external forces.
One can easily obtain the formulae analogous to (2.11)-(2.13) for ρ_α and $\mathbf{u}_\alpha$ in terms of B_α and $\mathbf{Z}_\alpha$. Hence, the Lagrangian (3.5) may be rewritten in the form (3.4). The variation associated with the application (3.1) yields

$$\delta_1 a\ =\ \int_{\Omega_1} \delta_1\ (L \det B_1)\ d\Omega_1.$$

Calculations presented in Appendix B give the following result:

$$\delta_1 a\ =\ \int_\Omega \left\{ tr \left(T\ \frac{\partial \boldsymbol{\zeta}_1}{\partial \mathbf{z}} - T_1\ \frac{\partial \delta_1 \mathbf{Z}_2}{\partial \mathbf{z}} \right)\ +\ \mathbf{S}_1^* \boldsymbol{\zeta}_1 \right\}\ d\Omega, \tag{3.6}$$

where

$$\begin{cases} T\ =\ L\ I + B_1\ \dfrac{\partial L}{\partial B_1} + B_2\ \dfrac{\partial L}{\partial B_2}, \\ T_1\ =\ B_2\ \dfrac{\partial L}{\partial B_2} B_2, \\ \mathbf{S}_1^*\ =\ \dfrac{\partial L}{\partial \mathbf{Z}_2} B_2^{-1} + \dfrac{\partial L}{\partial \mathbf{z}}. \end{cases} \tag{3.7}$$

The Gauss-Ostrogradskii formula and relation (3.3) involve

$$\delta_1 a \;=\; \int_\Omega \left\{ \mathbf{S}_1^* - \mathrm{Div}\ T + (\mathrm{Div}\ T_1) B_2^{-1} \right\} \boldsymbol{\zeta}_1\, d\Omega + \int_{\partial\Omega} \mathbf{N}^* (T - T_1 B_2^{-1})\ \boldsymbol{\zeta}_1\, d\Sigma. \quad (3.8)$$

Using the arguments described in section 2, we obtain from (3.8) both governing equations and Rankine-Hugoniot conditions for component "1"

$$\mathbf{S}_1^* - \mathrm{Div}\ T + (\mathrm{Div}\ T_1) B_2^{-1} = 0. \quad (3.9)$$

$$\mathbf{N}^* [T - T_1 B_2^{-1}] \;=\; 0. \quad (3.10)$$

Since $T - T_1 B_2^{-1} \;=\; L\ I + B_1\ \dfrac{\partial L}{\partial B_1}$, equation (3.10) is equivalent to

$$\mathbf{N}^* \left[\ L\ I + B_1\ \frac{\partial L}{\partial B_1}\ \right] \;=\; 0. \quad (3.11)$$

Equations for component "2" are obtained by permutation indexes "1" and "2":

$$\mathbf{S}_2^* - \mathrm{Div}\ T + (\mathrm{Div}\ T_2) B_1^{-1} \;=\; 0 \quad (3.9')$$

and

$$\mathbf{N}^* [T - T_2 B_1^{-1}] \;=\; 0 \quad (3.10')$$

Formula (3.10') can be rewritten in an equivalent form

$$\mathbf{N}^* \left[\ L\ I + B_2\ \frac{\partial L}{\partial B_2}\ \right] \;=\; 0. \quad (3.11')$$

Let us remark that the equations (3.9) and (3.9') are not in a divergence form. Let us denote

$$\mathbf{S}^* \;=\; \frac{\partial L}{\partial \mathbf{z}} \quad (3.12)$$

and

$$\mathbf{S}_{0\alpha}^* \;=\; \det\ B_\alpha\ \frac{\partial L}{\partial \mathbf{Z}_\alpha}\ ,\ T_{0\alpha} \;=\; -\det B_\alpha \frac{\partial L}{\partial B_\alpha} B_\alpha. \quad (3.13)$$

Identities

$$T_{02} \equiv -(\det B_2) B_2^{-1} T_1\ ,\ T_{01} \equiv -(\det B_1) B_1^{-1} T_2,$$

$$\mathbf{S}_1^* \equiv \mathbf{S}^* + \frac{1}{\det B_2}\ \mathbf{S}_{02}^* B_2^{-1}\ ,\ \mathbf{S}_2^* \equiv \mathbf{S}^* + \frac{1}{\det B_1}\ \mathbf{S}_{01}^* B_1^{-1},$$

and (1.1) involve that (3.9) and (3.9') are equivalent to

$$\mathbf{S}^* - \mathrm{Div}\ T + \frac{1}{\det B_1}\ (\mathbf{S}_{01}^* - \mathrm{Div}_1\ T_{01}) B_1^{-1} \;=\; 0, \quad (3.14)$$

and

$$\mathbf{S}^* - \mathrm{Div}\ T + \frac{1}{\det B_2}\ (\mathbf{S}_{02}^* - \mathrm{Div}_2\ T_{02}) B_2^{-1} \;=\; 0, \quad (3.14')$$

where $\mathrm{Div}_\alpha\ T_{0\alpha}$ means the divergence of $T_{0\alpha}$ with respect to the $\alpha-$th Lagrangian coordinates. The following identity $(C.1)$ proved in Appendix C,

$$\mathbf{S}^* - \mathrm{Div}\ T + \frac{1}{\det B_1}\ (\mathbf{S}_{01}^* - \mathrm{Div}_1\ T_{01}) B_1^{-1} \;+\; \frac{1}{\det B_2}\ (\mathbf{S}_{02}^* - \mathrm{Div}_2\ T_{02}) B_2^{-1} \equiv 0$$

and (3.14), (3.14') yield

$$\mathbf{S}^* - \mathrm{Div}\ T \;=\; 0. \quad (3.15)$$

We see in the next section that for fluid mixtures, equation (3.15) represents the conservation laws of total momentum and total energy. Hence, (3.14) and (3.14') are equivalent to:

$$\mathbf{S}^*_{01} - \mathrm{Div}_1 T_{01} \; = \; 0, \tag{3.16}$$

$$\mathbf{S}^*_{02} - \mathrm{Div}_2 T_{02} \; = \; 0. \tag{3.16$'$}$$

Equations (3.16) and (3.16') represent equations of motion for each component of a two-fluid medium in Lagrangian coordinates. They are less useful than equations in the Eulerian coordinates. However, they are in a divergence form and involve the conservation of the total momentum and the total energy.

4. Application to two-fluid mixtures

For a two-fluid mixture, the Lagrangian is (see (3.5)) :

$$L \; = \; \frac{1}{2}\rho_1|\mathbf{V}_1|^2 + \; \frac{1}{2}\rho_2|\mathbf{V}_2|^2 - W(\rho_1,\rho_2,s_1,s_2,\mathbf{w}) - \rho\Pi(\mathbf{z}), \tag{4.1}$$

where

$$\mathbf{V}_\alpha \; = \; \begin{pmatrix} 1 \\ \mathbf{u}_\alpha \end{pmatrix} \equiv \; \frac{B_\alpha}{\mu_\alpha}\, \boldsymbol{\ell}\;, \; \begin{pmatrix} 0 \\ \mathbf{w} \end{pmatrix} \; = \; \mathbf{V}_2 - \mathbf{V}_1. \tag{4.2}$$

$$\rho_\alpha \; = \; \frac{\mu_\alpha \; \rho_{0\alpha}\;(\mathbf{X}_\alpha)}{\det B_\alpha}, \tag{4.3}$$

$$s_\alpha \; = \; s_{0\alpha}(\mathbf{Z}_\alpha). \tag{4.4}$$

Definitions (4.1)-(4.4) involve the governing equations (3.9), (3.9') in the following form (see Appendix D):

$$\left\{\begin{array}{l} \dfrac{\partial \mathbf{K}_\alpha}{\partial t} + \mathrm{rot}\;\mathbf{K}_\alpha \times \; \mathbf{u}_\alpha = \nabla^*(\mathbf{R}_\alpha - \mathbf{K}^*_\alpha \mathbf{u}_\alpha) + \theta_\alpha \nabla^* s_\alpha, \\ \dfrac{\partial \rho_\alpha}{\partial t} + \mathrm{div}\;(\rho_\alpha \mathbf{u}_\alpha) \; = \; 0, \\ \dfrac{\partial (\rho_\alpha s_\alpha)}{\partial t} + \mathrm{div}\;(\rho_\alpha s_\alpha \mathbf{u}_\alpha) \; = \; 0, \end{array}\right. \tag{4.5}$$

where

$$\left\{\begin{array}{l} \rho_\alpha \mathbf{K}^*_\alpha \equiv \; \dfrac{\partial L}{\partial \mathbf{u}_\alpha} = \rho_\alpha \mathbf{u}^*_\alpha - (-1)^\alpha \; \dfrac{\partial W}{\partial \mathbf{w}}\;, \quad \text{with} \;\; \mathbf{w} = \mathbf{u}_2 - \mathbf{u}_1, \\ R_\alpha \equiv \; \dfrac{\partial L}{\partial \rho_\alpha} = \dfrac{1}{2}|\mathbf{u}_\alpha|^2 - \dfrac{\partial W}{\partial \rho_\alpha} - \Pi, \\ \rho_\alpha \theta_\alpha \equiv -\dfrac{\partial L}{\partial s_\alpha} = \dfrac{\partial W}{\partial s_\alpha}. \end{array}\right.$$

We notice here that the governing equations were obtained earlier by a different method in [11]. Equations (4.5) yield the conservation laws for the total momentum and the total energy associated with (3.15)

$$\frac{\partial}{\partial t}(\rho_1 \mathbf{K}^*_1 + \rho_2 \mathbf{K}^*_2) +$$

$$+\mathrm{div}\left(\rho_1 \mathbf{u}_1 \mathbf{K}^*_1 + \rho_2 \mathbf{u}_2 \mathbf{K}^*_2 + (\rho_1 \; \frac{\partial W}{\partial \rho_2} + \rho_2 \frac{\partial W}{\partial \rho_2} - W)\; I\right) + \rho \nabla \Pi = 0,$$

$$\frac{\partial}{\partial t}\Big(\sum_{\alpha=1}^{2}\rho_\alpha\ \frac{1}{2}|\mathbf{u}_\alpha|^2+\rho\Pi+W-\frac{\partial W}{\partial \mathbf{w}}\mathbf{w}\Big)+$$

$$+\mathrm{div}\ \Big(\sum_{\alpha=1}^{2}\rho_\alpha\mathbf{u}_\alpha(\mathbf{K}_\alpha^*\mathbf{u}_\alpha-R_\alpha)\Big)-\rho\ \frac{\partial\Pi}{\partial t}\ =0. \tag{4.6}$$

In the general case, they are the only conservation laws admitted by the system. Hence, the system (4.5) is not in a divergence form. The Rankine -Hugoniot conditions (3.10)-(3.10') (or (3.11)-(3.11')) for this system are obtained in Appendix A(b) (see formulae (A.9)); they are

$$\Big[-D_n(L-\rho_\alpha\mathbf{K}_\alpha^*\mathbf{u}_\alpha)\ +\ \rho_\alpha\mathbf{n}^*\ \mathbf{u}_\alpha(R_\alpha-\mathbf{K}_\alpha^*\mathbf{u}_\alpha)\Big]\ =\ 0, \tag{4.7^1}$$

$$\Big[-D_n\rho_\alpha\mathbf{K}_\alpha^*+\mathbf{n}^*\Big((L-\rho_\alpha R_\alpha)\ I+\rho_\alpha\mathbf{u}_\alpha\mathbf{K}_\alpha^*\Big)\Big]\ =\ 0. \tag{4.7^2}$$

In addition, the mass conservations laws are expressed in the form

$$\Big[\rho_\alpha(\mathbf{n}^*\mathbf{u}_\alpha-D_n)\Big]\ =\ 0. \tag{4.7^3}$$

Let us consider **shock waves** when $\mathbf{n}^*\mathbf{u}_\alpha-D_n\ \neq\ 0$. Taking into account (4.7^2) and (4.7^3) we obtain

$$\Big[\mathbf{K}_\alpha-(\mathbf{K}_\alpha^*\mathbf{n})\mathbf{n}\Big]\ =\ 0, \tag{4.8^1}$$

which means that $[\mathbf{K}_\alpha]$ is normal to the shock. By using (4.7^1) and the identity

$$\Big[L-\rho_\alpha\mathbf{K}_\alpha^*\mathbf{u}_\alpha\Big]\ =\ \frac{1}{D_n}\ \mathbf{n}^*\Big[\rho_\alpha\mathbf{u}_\alpha(R_\alpha-\mathbf{K}_\alpha^*\mathbf{u}_\alpha)\Big],$$

we obtain from relation (4.7^2)

$$\Big[\mathbf{K}_\alpha^*\mathbf{u}_\alpha-R_\alpha-D_n(\mathbf{K}_\alpha^*\mathbf{n})\Big]\ =\ 0. \tag{4.8^2}$$

Consequently, (4.7^1) yields to

$$\Big[L-\rho_\alpha R_\alpha+\rho_\alpha(\mathbf{n}^*\mathbf{u}_\alpha-D_n)\mathbf{K}_\alpha^*\mathbf{n}\Big]\ =\ 0. \tag{4.8^3}$$

Equations (4.7^3), (4.8^1) - (4.8^3) form a complete set of eight scalar Rankine-Hugoniot conditions, representing the conservation of mass, momentum and energy of α-th component. For the gas dynamics, these conditions correspond to the classical shock conditions. We have to emphasize here that for the two-fluid model equations (4.8^1) $-$ (4.8^3) do not involve the conservation of the total momentum and the total energy through the shock. We notice that the jump conditions (4.8^1) $-$ (4.8^2) were obtained earlier for barotropic fluids using the variations in the Eulerian coordinates [1-2].
For **contact discontinuities**, when $\mathbf{n}^*\mathbf{u}_\alpha-D_n\ =\ 0$, we get from (4.7^1) $-$ (4.7^3)

$$\Big[L-\rho_\alpha R_\alpha\Big]\ =\ 0. \tag{4.8^4}$$

For the gas dynamics equation (4.8^4) corresponds to the continuity of the pressure. All the jump conditions are obtained from the Hamilton principle without any ambiguity.

5. Conclusion

Using the Hamilton principle we have obtained in the general conservative case the governing equations for two-fluid mixtures (4.5). The equations for the total quantities (total momentum and total energy) are in a divergence form (see (4.6)). The equations for the components are in divergence form only in the Lagrangian coordinates (see (3.16)-(3.16')). The Hamilton principle gives also a set of Rankine-Hugoniot conditions $(4.7^1) - (4.7^2)$. Together with the equations of mass (4.7^3) they form a complete set of the jump relations. For the gas dynamics model they coincide with the classical Rankine-Hugoniot conditions of conservation of mass, momentum and energy.

6. Appendix A

In gas dynamics, the Lagrangian (2.9) is a function of $\mathbf{V} = \begin{pmatrix} 1 \\ \mathbf{u} \end{pmatrix}$, ρ, s and $\mathbf{z}$. Formulae (2.11)-(2.13) allow us to consider the Lagrangian as a function of $\mathbf{z}, B, \mathbf{Z}$:

$$\begin{array}{c} \mathbf{V} = \dfrac{B}{\mu}\,\boldsymbol{\ell}\ ,\ \mu = \boldsymbol{\ell}^* B \boldsymbol{\ell}, \\ \rho\ \det\ B = \mu\ \rho_0(\mathbf{X}),\ s = s_0(\mathbf{Z}) \end{array} . \qquad (A.1)$$

We need to calculate $T = L\ I + B\ \dfrac{\partial L}{\partial B}$. From (A.1) we have:

$$d\mu = \boldsymbol{\ell}^*\ dB\ \boldsymbol{\ell} = tr(\boldsymbol{\ell}\boldsymbol{\ell}^*\ dB), \qquad (A.2)$$

$$d\mathbf{V} = \frac{dB}{\mu}\,\boldsymbol{\ell} - \frac{B\boldsymbol{\ell}}{\mu^2}\,d\mu = \frac{dB}{\mu}\,\boldsymbol{\ell} - \frac{B\boldsymbol{\ell}}{\mu^2}\,tr\ (\boldsymbol{\ell}\boldsymbol{\ell}^* dB), \qquad (A.3)$$

$$\frac{\partial\rho}{\partial B} = \rho_0(\mathbf{X})\mu\frac{\partial}{\partial B}\left(\frac{1}{\det B}\right) + \rho_0(\mathbf{X})\frac{1}{\det B}\frac{\partial\mu}{\partial B} = \rho\left(\frac{\boldsymbol{\ell}\boldsymbol{\ell}^*}{\mu} - B^{-1}\right). \qquad (A.4)$$

Moreover,

$$\frac{\partial s}{\partial B} = 0. \qquad (A.5)$$

The differential of $L = L(\mathbf{V}, \rho, s, \mathbf{z})$ is:

$$dL = \frac{\partial L}{\partial \mathbf{V}}\,d\mathbf{V} + \frac{\partial L}{\partial \rho}\,d\rho + \frac{\partial L}{\partial s}\,ds + \frac{\partial L}{\partial \mathbf{z}}\,d\mathbf{z}.$$

Taking into account relations(A.2)-(A.5), we obtain

$$\begin{aligned} dL = {} & \frac{\partial L}{\partial \mathbf{V}}\left(\frac{dB\ \boldsymbol{\ell}}{\mu} - \frac{B\boldsymbol{\ell}}{\mu^2}\,tr(\boldsymbol{\ell}\boldsymbol{\ell}^* dB)\right) + \\ & + \frac{\partial L}{\partial \rho}\,d\rho + \frac{\partial L}{\partial s}\,ds + \frac{\partial L}{\partial \mathbf{z}}\,d\mathbf{z} = \end{aligned}$$

$$tr\left(\frac{\boldsymbol{\ell}}{\mu}\frac{\partial L}{\partial \mathbf{V}}dB-\frac{\frac{\partial L}{\partial \mathbf{V}}B\boldsymbol{\ell}}{\mu^2}\boldsymbol{\ell}\boldsymbol{\ell}^*dB\right)+$$

$$+\frac{\partial L}{\partial \rho}tr\left(\frac{\partial \rho}{\partial B}\,dB\right)+\left(\frac{\partial L}{\partial \rho}\frac{\partial \rho}{\partial \mathbf{Z}}+\frac{\partial L}{\partial s}\frac{\partial s}{\partial \mathbf{Z}}\right)d\mathbf{Z}+\frac{\partial L}{\partial \mathbf{z}}d\mathbf{z}.$$

Hence,

$$\frac{\partial L}{\partial B}=\frac{\boldsymbol{\ell}\frac{\partial L}{\partial \mathbf{V}}}{\mu}-\frac{\boldsymbol{\ell}\boldsymbol{\ell}^*\left(\frac{\partial L}{\partial \mathbf{V}}B\boldsymbol{\ell}\right)}{\mu^2}+\frac{\partial L}{\partial \rho}\frac{\partial \rho}{\partial B}. \qquad (A.6)$$

Then

$$B\frac{\partial L}{\partial B}=\frac{B\boldsymbol{\ell}\frac{\partial L}{\partial \mathbf{V}}}{\mu}-\frac{\left(\left(\frac{\partial L}{\partial \mathbf{V}}B\boldsymbol{\ell}\right)B\boldsymbol{\ell}\boldsymbol{\ell}^*\right)}{\mu^2}+\rho\frac{\partial L}{\partial \rho}\left(B\frac{\boldsymbol{\ell}\boldsymbol{\ell}^*}{\mu}-\ I\right).$$

Taking into account (A.1), we obtain

$$B\frac{\partial L}{\partial B}=\mathbf{V}\frac{\partial L}{\partial \mathbf{V}}-\left(\frac{\partial L}{\partial \mathbf{V}}\mathbf{V}\right)\mathbf{V}\boldsymbol{\ell}^*+\rho\frac{\partial L}{\partial \rho}(\mathbf{V}\boldsymbol{\ell}^*-\ I) \qquad (A.7)$$

and, (A.7) yields then

$$T=L\ I+B\frac{\partial L}{\partial B}=\left(L-\rho\frac{\partial L}{\partial \rho}\right)\ I+\mathbf{V}\frac{\partial L}{\partial \mathbf{V}}-\left(\frac{\partial L}{\partial \mathbf{V}}\mathbf{V}-\rho\frac{\partial L}{\partial \rho}\right)\mathbf{V}\boldsymbol{\ell}^*. \qquad (A.8)$$

a) In the case $L=\frac{1}{2}\rho\mathbf{V}^*\mathbf{V}-\varepsilon(\rho,s)-\rho\Pi$ formula (A.8) can be rewritten in the form

$$T=p\ I+\rho\mathbf{V}\mathbf{V}^*-\left(\frac{1}{2}\mathbf{V}^*\mathbf{V}+\varepsilon'_\rho+\Pi\right)\rho\mathbf{V}\boldsymbol{\ell}^*,$$

where $p=\rho\varepsilon'_\rho-\varepsilon$. Then,

$$T=\begin{bmatrix} -E\ ,\ \rho\mathbf{u}^* \\ \\ -(E+p)\mathbf{u}\ ,\ p\ I+\rho\mathbf{u}\mathbf{u}^* \end{bmatrix},$$

where $E=\frac{1}{2}\rho\mathbf{V}^*\mathbf{V}+\varepsilon+\rho\Pi$ is the total volume energy.
Moreover, since $\mathbf{S}^*=-\rho\frac{\partial \Pi}{\partial \mathbf{z}}$, we get the energy and momentum equations for gas dynamics motions

$$\begin{cases} \dfrac{\partial E}{\partial t}+\ \mathrm{div}\ \Big((E+p)\mathbf{u}\Big)-\rho\dfrac{\partial \Pi}{\partial t}=0, \\ \dfrac{\partial \rho\mathbf{u}^*}{\partial t}+\ \mathrm{div}\ (p\ I+\rho\mathbf{u}\mathbf{u}^*)+\rho\dfrac{\partial \Pi}{\partial \mathbf{x}}=0. \end{cases}$$

Because $\mathbf{N}^*$ is collinear to $(\ -D_n,\ \ \mathbf{n}^*\ )$, the classical Rankine-Hugoniot conditions (2.15) are

$$\Big[D_nE-(E+p)\mathbf{n}^*\mathbf{u}\Big]\ =\ 0,$$

$$\Big[D_n\rho\ \mathbf{u}^*-\mathbf{n}^*(p\ I+\rho\mathbf{u}\mathbf{u}^*)\Big]\ =\ 0.$$

b) For the two-fluid model the Lagrangian is

$$L=\frac{1}{2}\rho_1\mathbf{V}_1^*\mathbf{V}_1+\frac{1}{2}\rho_2\mathbf{V}_2^*\mathbf{V}_2-W(\rho_1,\rho_2,s_1,s_2,\mathbf{w})$$

with

$$\begin{pmatrix} 0 \\ \mathbf{w} \end{pmatrix} = \begin{pmatrix} 0 \\ \mathbf{u}_2 - \mathbf{u}_1 \end{pmatrix} = \mathbf{V}_2 - \mathbf{V}_1.$$

We obtain similarly to (A.8)

$$L\ I + B_\alpha \frac{\partial L}{\partial B_\alpha} = \left(L - \rho_\alpha \frac{\partial L}{\partial \rho_\alpha} \right)\ I + \mathbf{V}_\alpha \frac{\partial L}{\partial \mathbf{V}_\alpha} - \left(\frac{\partial L}{\partial \mathbf{V}_\alpha} \mathbf{V}_\alpha - \rho_\alpha \frac{\partial L}{\partial \rho_\alpha} \right) \mathbf{V}_\alpha \boldsymbol{\ell}^*.$$

Here

$$\frac{\partial L}{\partial \mathbf{V}_\alpha} = \rho_\alpha \mathbf{V}_\alpha^* - \frac{\partial W}{\partial \mathbf{V}_\alpha} = \rho_\alpha \mathbf{V}_\alpha^* - \frac{\partial W}{\partial \mathbf{w}} \frac{\partial \mathbf{w}}{\partial \mathbf{V}_\alpha} \equiv \rho_\alpha (1, \mathbf{K}_\alpha^*),$$

$$\frac{\partial L}{\partial \rho_\alpha} = \frac{1}{2} \mathbf{V}_\alpha^* \mathbf{V}_\alpha - \frac{\partial W}{\partial \rho_\alpha} - \Pi \equiv R_\alpha.$$

In a matrix form, we have

$$L\ I + B_\alpha \frac{\partial L}{\partial B_\alpha} \quad = \quad \begin{bmatrix} L - \rho_\alpha \mathbf{K}_\alpha^* \mathbf{u}_\alpha\ ,\ \rho_\alpha \mathbf{K}_\alpha^* \\ \\ -\rho_\alpha \mathbf{u}_\alpha \mathbf{K}_\alpha^* \mathbf{u}_\alpha + \rho_\alpha R_\alpha \mathbf{u}_\alpha\ ,\ (L - \rho_\alpha R_\alpha)\ I + \rho_\alpha \mathbf{u}_\alpha \mathbf{K}_\alpha^* \end{bmatrix},$$

and the Rankine-Hugoniot conditions for the two-fluid model are

$$\mathbf{N}^* \begin{bmatrix} L - \rho_\alpha \mathbf{K}_\alpha^* \mathbf{u}_\alpha\ ,\ \rho_\alpha \mathbf{K}_\alpha^* \\ \\ \rho_\alpha \mathbf{u}_\alpha (R_\alpha - \mathbf{K}_\alpha^* \mathbf{u}_\alpha)\ ,\ (L - \rho_\alpha R_\alpha)\ I + \rho_\alpha \mathbf{u}_\alpha \mathbf{K}_\alpha^* \end{bmatrix} \ = \ 0. \qquad (A.9)$$

7. Appendix B

The variation $\delta_1 a$ is:

$$\delta_1 a = \int_\Omega \left\{ \delta_1 L + L\ \delta_1 (\det B_1)(\det B_1)^{-1} \right\} d\Omega = \int_\Omega \left\{ \delta_1 L + tr \left(L \frac{\partial \boldsymbol{\zeta}_1}{\partial z} \right) \right\} d\Omega.$$

Formula (3.3) involves

$$\delta_1 L = tr \left\{ \frac{\partial L}{\partial B_1} \delta_1 B_1 + \frac{\partial L}{\partial B_2} \delta_1 B_2 \right\} + \frac{\partial L}{\partial \mathbf{Z}_2} B_2^{-1} \boldsymbol{\zeta}_1 + \frac{\partial L}{\partial \mathbf{z}} \boldsymbol{\zeta}_1. \qquad (B.1)$$

By definition,

$$\delta_1 B_1 = \frac{\partial \boldsymbol{\zeta}_1}{\partial \mathbf{z}} B_1. \qquad (B.2)$$

Let us calculate $\delta_1 B_2$. We get

$$\delta_1 B_2 = \delta_1 \left(\frac{\partial \mathbf{z}}{\partial \mathbf{Z}_2} \right) \ = \ \delta_1 \left(\frac{\partial \mathbf{z}}{\partial \mathbf{Z}_1} \frac{\partial \mathbf{Z}_1}{\partial \mathbf{Z}_2} \right) = \delta_1 \left(\frac{\partial \mathbf{z}}{\partial \mathbf{Z}_1} \right) \frac{\partial \mathbf{Z}_1}{\partial \mathbf{Z}_2} + \frac{\partial \mathbf{z}}{\partial \mathbf{Z}_1} \delta_1 \left(\frac{\partial \mathbf{Z}_1}{\partial \mathbf{Z}_2} \right). \quad (B.3)$$

Since for any linear mapping $A : \delta(A^{-1}) = -A^{-1} \delta A\ A^{-1}$, we have

$$\delta_1 \left(\frac{\partial \mathbf{Z}_1}{\partial \mathbf{Z}_2} \right) = -\frac{\partial \mathbf{Z}_1}{\partial \mathbf{Z}_2} \delta_1 \left(\frac{\partial \mathbf{Z}_2}{\partial \mathbf{Z}_1} \right) \frac{\partial \mathbf{Z}_1}{\partial \mathbf{Z}_2} = -\frac{\partial \mathbf{Z}_1}{\partial \mathbf{Z}_2} \frac{\partial \delta_1 \mathbf{Z}_2}{\partial \mathbf{Z}_1} \frac{\partial \mathbf{Z}_1}{\partial \mathbf{Z}_2}. \qquad (B.4)$$

Formulae (B.3) and (B.4) involve that

$$\delta_1 B_2 = \frac{\partial \boldsymbol{\zeta}_1}{\partial \mathbf{z}} B_2 - \frac{\partial \mathbf{z}}{\partial \mathbf{Z}_2} \frac{\partial \delta_1 \mathbf{Z}_2}{\partial \mathbf{Z}_2} = \frac{\partial \boldsymbol{\zeta}_1}{\partial \mathbf{z}} B_2 - B_2 \frac{\partial \delta_1 \mathbf{Z}_2}{\partial \mathbf{z}} B_2. \qquad (B.5)$$

Substituting (B.2) and (B.5) into (B.1) we get

$$\delta_1 L = tr\left(\frac{\partial L}{\partial B_1}\frac{\partial \boldsymbol{\zeta}_1}{\partial \mathbf{z}}B_1 + \frac{\partial L}{\partial B_2}\left(\frac{\partial \boldsymbol{\zeta}_1}{\partial \mathbf{z}}B_2 - B_2\frac{\partial \delta_1 \mathbf{Z}_2}{\partial \mathbf{z}}B_2\right)\right) + \frac{\partial L}{\partial \mathbf{Z}_2}B_2^{-1}\boldsymbol{\zeta}_1 + \frac{\partial L}{\partial \mathbf{z}}\boldsymbol{\zeta}_1 =$$

$$= tr\left\{\left(B_1\frac{\partial L}{\partial B_1} + B_2\frac{\partial L}{\partial B_2}\right)\frac{\partial \boldsymbol{\zeta}_1}{\partial \mathbf{z}} - B_2\frac{\partial L}{\partial B_2}B_2\frac{\partial \delta_1 \mathbf{Z}_2}{\partial \mathbf{z}}\right\} + \frac{\partial L}{\partial \mathbf{Z}_2}B_2^{-1}\boldsymbol{\zeta}_1 + \frac{\partial L}{\partial \mathbf{z}}\boldsymbol{\zeta}_1.$$

Hence,

$$\delta_1 L + tr\left(L\frac{\partial \boldsymbol{\zeta}_1}{\partial \mathbf{z}}\right) = tr\left(T\frac{\partial \boldsymbol{\zeta}_1}{\partial \mathbf{z}} - T_1\frac{\partial \delta_1 \mathbf{Z}_2}{\partial \mathbf{z}}\right) + \mathbf{S}_1^*\boldsymbol{\zeta}_1,$$

where T , T_1 and $\mathbf{S}_1^*$ are defined in (3.7). The relation (3.6) is proved.

8. Appendix C

Theorem: *The following expression is an identity*

$$\mathbf{S}^* - \text{Div}\, T + \frac{1}{\det B_1}\left(\mathbf{S}_{01}^* - \text{Div} T_{01}\right) B_1^{-1} + \frac{1}{\det B_2}\left(\mathbf{S}_{02}^* - \text{Div}\, T_{02}\right) B_2^{-1} \equiv 0, \quad (C.1)$$

where (see formulae (3.6), (3.12) -(3.13))

$$\begin{cases} \mathbf{S}^* = \dfrac{\partial L}{\partial \mathbf{z}}, \\ T = L\, I + B_1\dfrac{\partial L}{\partial B_1} + B_2\dfrac{\partial L}{\partial B_2}, \\ \mathbf{S}_{0\alpha}^* = \det B_\alpha\dfrac{\partial L}{\partial \mathbf{Z}_\alpha}, \\ T_{0\alpha} = -\det B_\alpha\dfrac{\partial L}{\partial B_\alpha}B_\alpha. \end{cases}$$

Proof: Using (1.1), we get

$$\sum_{\alpha=1}^{2}\frac{1}{\det B_\alpha}(\mathbf{S}_{0\alpha}^* - \text{Div}_\alpha T_{0\alpha})B_\alpha^{-1} \equiv \sum_{\alpha=1}^{2}\left(\frac{\partial L}{\partial \mathbf{Z}_\alpha} + \text{Div}(B_\alpha\frac{\partial L}{\partial B_\alpha}B_\alpha)\right)B_\alpha^{-1}.$$

We have to prove that this expression is identical to

$$\text{Div}\, T - \mathbf{S}^* \equiv \text{Div}\left(L\, I + B_1\frac{\partial L}{\partial B_1} + B_2\frac{\partial L}{\partial B_2}\right) - \frac{\partial L}{\partial \mathbf{z}}.$$

Indeed, by definition

$$\text{Grad}\, L \equiv \frac{\partial L}{\partial \mathbf{z}} + \frac{\partial L}{\partial \mathbf{Z}_1}\frac{\partial \mathbf{Z}_1}{\partial \mathbf{z}} + \frac{\partial L}{\partial \mathbf{Z}_2}\frac{\partial \mathbf{Z}_2}{\partial \mathbf{z}} + tr\left(\frac{\partial L}{\partial B_1}\frac{\partial B_1}{\partial \mathbf{z}}\right) + tr\left(\frac{\partial L}{\partial B_2}\frac{\partial B_2}{\partial \mathbf{z}}\right),$$

where

$$\left(tr\left(\frac{\partial L}{\partial B_\alpha}\frac{\partial B_\alpha}{\partial \mathbf{z}}\right)\right)_p \equiv \left(\frac{\partial L}{\partial B_\alpha}\right)_i^j\frac{\partial B_j^i}{\partial z^p}.$$

Hence, we have to verify only the formula

$$\text{Div}\left(B_\alpha\frac{\partial L}{\partial B_\alpha}B_\alpha\right)B_\alpha^{-1} \equiv \text{Div}\left(B_\alpha\frac{\partial L}{\partial B_\alpha}\right) + tr\left(\frac{\partial L}{\partial B_\alpha}\frac{\partial B_\alpha}{\partial \mathbf{z}}\right). \qquad (C.2)$$

To prove the identity (C.2) we may consider only $L = L(B)$, with $B = \dfrac{\partial \mathbf{z}}{\partial \mathbf{Z}}$.
Let $A = \dfrac{\partial L}{\partial B}$, then

$$\frac{\partial L}{\partial z^m} = tr\left(\frac{\partial L}{\partial B}\,\frac{\partial B}{\partial z^m}\right) = a^i_j\,\frac{\partial b^j_i}{\partial z^m}.$$

Hence,

$$\frac{\partial (b^i_j\ a^j_k\ b^k_s)}{\partial z^i}\ \bar b^s_p = \frac{\partial (b^i_j\quad a^j_k)}{\partial z^i}\ b^k_s\ \bar b^s_p + b^i_j\ a^j_k\ \frac{\partial b^k_s}{\partial z^i}\ \bar b^s_p =$$
$$\frac{\partial (b^i_j\ a^j_p)}{\partial z^i} + b^i_j\ a^j_k\ \frac{\partial b^k_s}{\partial Z^m}\ \frac{\partial Z^m}{\partial z^i}\ \bar b^s_p = \frac{\partial (b^i_j\ a^j_p)}{\partial z^i} + b^i_j\ \bar b^m_i\ a^j_k\ \frac{\partial b^k_s}{\partial Z^m}\bar b^s_p =$$
$$\frac{\partial (b^i_j\ a^j_p)}{\partial z^i} + a^m_k\ \frac{\partial b^k_m}{\partial Z^s}\bar b^s_p = \frac{\partial (b^i_j\ a^j_p)}{\partial z^i} + a^m_k\ \frac{\partial b^k_m}{\partial z^l}\ b^l_s \bar b^s_p = \frac{\partial (b^i_j\ a^j_p)}{\partial z^i} + a^m_k\ \frac{\partial b^k_m}{\partial z^p}.$$

This is a proof of (C.2) and consequently (C.1).

9. Appendix D

First of all, we obtain the governing equations for each component in the Lagrangian coordinates. These equations yield easily the governing equation in the Eulerian coordinates. In the Lagrangian coordinates equations (3.16) – (3.16$'$) are

$$\mathbf{S}^*_{0\alpha} - \mathrm{Div}_\alpha T^*_{0\alpha} = 0,$$

where

$$\mathbf{S}^*_{0\alpha} = \det B_\alpha\ \frac{\partial L}{\partial \mathbf{Z}_\alpha}\ = R_\alpha \mu_\alpha\ \frac{\partial \rho_{0\alpha}}{\partial \mathbf{Z}_\alpha} - det B_\alpha\ \rho_\alpha \theta_\alpha \frac{\partial s_{0\alpha}}{\partial \mathbf{Z}_\alpha},$$

and

$$T_{0\alpha} = -\det B_\alpha\ \frac{\partial L}{\partial B_\alpha} B_\alpha.$$

Similarly to (A.6) we obtain

$$\frac{\partial L}{\partial B_\alpha} = \frac{\rho_\alpha \boldsymbol{\ell}\frac{\partial L}{\partial \mathbf{V}_\alpha}}{\mu_\alpha} - \frac{\boldsymbol{\ell}\boldsymbol{\ell}^*}{\mu_\alpha}(\rho_\alpha \frac{\partial L}{\partial \mathbf{V}_\alpha}\mathbf{V}_\alpha) + \rho_\alpha R_\alpha \left(\frac{\boldsymbol{\ell}\boldsymbol{\ell}^*}{\mu_\alpha} - B^{-1}_\alpha\right).$$

Consequently,

$$\det B_\alpha\ \frac{\partial L}{\partial B_\alpha} B_\alpha = \rho_{0\alpha}\ \boldsymbol{\ell}\frac{\partial L}{\partial \mathbf{V}_\alpha} B_\alpha - \rho_{0\alpha}\ (\frac{\partial L}{\partial \mathbf{V}_\alpha}\mathbf{V}_\alpha)\boldsymbol{\ell}\boldsymbol{\ell}^* B_\alpha + \rho_{0\alpha} R_\alpha(\boldsymbol{\ell}\boldsymbol{\ell}^* B_\alpha - \mu_\alpha\ I).$$

If $\lambda_\alpha = t,\ \ \mu_\alpha = 1, B_\alpha = \begin{bmatrix} 1 & 0 \\ \mathbf{u}_\alpha & F_\alpha \end{bmatrix}$ and consequently,

$$T_{0\alpha} = \rho_{0\alpha} \begin{bmatrix} 0 & , & -\mathbf{K}^*_\alpha F_\alpha \\ & & \\ 0 & , & R_\alpha\ I \end{bmatrix}.$$

The governing equations of α-th component in the Lagrangian coordinates (t, X_α) are:

$$-\rho_\alpha \theta_\alpha \det B_\alpha\ \frac{\partial s_{0\alpha}}{\partial t} = 0,$$

$$\frac{\partial}{\partial t}(\rho_{0\alpha}\mathbf{K}^*_\alpha F_\alpha) -\ \mathrm{div}_\alpha(\rho_{0\alpha} R_\alpha\ I) + R_\alpha\ \frac{\partial \rho_{0\alpha}}{\partial \mathbf{X}_\alpha} - \rho_{0\alpha}\theta_\alpha\ \frac{\partial s_{0\alpha}}{\partial \mathbf{X}_\alpha}\ \ =\ 0.$$

Therefore,

$$\frac{\partial s_{0\alpha}}{\partial t} = 0, \qquad (D.1)$$

$$\frac{\partial}{\partial t}(\mathbf{K}^*_\alpha F_\alpha) - \nabla_\alpha R_\alpha - \theta_\alpha \nabla_\alpha \ s_{0\alpha} \ = \ 0. \qquad (D.2)$$

Taking into account the identity

$$\frac{\partial F_\alpha}{\partial t} = \frac{\partial \mathbf{u}_\alpha}{\partial \mathbf{x}} F_\alpha$$

and substituting the partial derivative with respect to time in the Lagrangian coordinates by the material derivative in the Eulerian coordinates, we obtain

$$\frac{d_\alpha}{dt} s_\alpha = 0, \quad \text{with} \quad \frac{d_\alpha}{dt} = \frac{\partial}{\partial t} \ + \ \mathbf{u}_\alpha{}^* \nabla^*, \qquad (D.1')$$

$$\frac{d_\alpha}{dt} \ \mathbf{K}^*_\alpha + \mathbf{K}^*_\alpha \ \frac{\partial \mathbf{u}_\alpha}{\partial \mathbf{x}} = \nabla R_\alpha + \theta_\alpha \nabla s_\alpha. \qquad (D.2')$$

Let us note that (D.1'), (D.2') and the mass conservation laws are equivalent to (4.5).

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